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THE α-UNION THEOREM AND GENERALIZED PRIMITIVE RECURSION(1)

BY

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ABSTRACT. A generalization to α-recursion theory of the McCreight-Meyer Union Theorem is proved. THEOREM. *Let Φ be an α-computational complexity measure and $\{f_\varepsilon | \varepsilon < \alpha\}$ an α-r.e. strictly increasing sequence of α-recursive functions. Then there exists an α-recursive function k such that* $C_k^\Phi = \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi$. The proof entails a no-injury cancellation atop a finite-injury priority construction and necessitates a blocking strategy to insure proper convergence.

Two infinite analogues to (ω-) primitive recursive functions are studied. Although these generalizations coincide at ω, they diverge on all admissible $\alpha > \omega$. Several well-known complexity properties of primitive recursive functions hold for one class but fail for the other. It is seen that the Jensen-Karp ordinally primitive recursive functions restricted to admissible $\alpha > \omega$ cannot possess natural analogues to Grzegorczyk's hierarchy.

0. Introduction. The motivation for the study of computation on infinite ordinals stems from several areas of mathematical logic. Takeuti [26], [27] was concerned with the problem of the reduction of the consistency of set theory to that of a theory of ordinal numbers. Machover [18], seeking to generalize model and recursion theoretic concepts to infinitary languages, developed a recursion theory on regular infinite cardinals. Questions of definability and their relation to higher logics and languages moved Kreisel [11], and later Kreisel and Sacks [12], to develop a recursion theory on Church and Kleene's ω^{ck}, called metarecursion theory.

From a set theoretic point of view, Jensen and Karp [9] developed the notion of an ordinally primitive recursive function (Prim_0). Karp's motivation came from an investigation into the classification of infinitary languages, Jensen's from the study of levels of Gödel's constructible hierarchy. A key

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result was a general bounding phenomenon (Stability Theorem) of Prim_0 functions on the primitive recursive set functions.

Kripke [13] (and independently Platek [20]) arrived at a unifying concept for the aforementioned cases; namely, the notion of admissible ordinal. The theory of computation on admissible ordinals α became known as α-recursion theory. By setting up an equation calculus (similar to Kleene (cf. [10]) for $\alpha = \omega$), Kripke was able to develop enough α-recursion theory to establish an infinite analogue to Kleene's T predicate and subsequently a Normal Form Theorem. From this he was able to assert that all of the results of unrelativized ordinary recursion theory (as found in Kleene [10]) hold in α-recursion theory.

There is strong interaction between the Jensen-Karp primitive recursive ordinal functions and α-recursion theory. Specifically, for admissible α, the closure of Prim_0 under a regular (cf. [3, p. 38]) α-bounded min operator yields the α-recursive functions. Further, the construction of analogues to Kleene's T predicate for α-recursion theory may be developed from only Prim_0 relations and functions, independent of α.

Deep results of ordinary recursion theory concern the notion of relativization and often require use of the powerful technique called the priority argument. Sacks and Simpson [21] introduced the priority method into α-recursion theory in their α-analogue to the Friedberg-Muchnik solution to Post's problem. Since then Sacks' students and coworkers have successfully demonstrated that major priority argument results generalize to α. The proofs of these results generally require vast modification from the ω situation in both the construction and their verifications. (See Lerman [16], Shore [23], Leggett and Shore [15].) An excellent survey can be found in Shore [25].

Another subarea of ordinary recursion theory, abstract complexity theory, has its major theorems obtained in a manner similar to that of relativized recursion theory. Founded upon several axioms of measure of complexity of computation (cf. [1]), deep results are established through constructions also based upon priority mechanisms. (See [1], [2], [19] and [29].) It was shown in [7] and [8] that the major results of this area generalize to α-recursion theory.

Many recursion theorists, however, are quick to point out differences between the two types of constructions. Consequently, they contend that the name "priority argument" be reserved only for the former type. Specifically, the former generally involves multiple (priority based) cancellations or injuries, the latter single (sometimes double) no-injury cancellations. However, an interesting phenomenon occurs upon generalizing the McCreight-Meyer Union Theorem [19] to α-recursion theory. While the ω-proof employs a typical no-injury cancellation construction, its lift to α requires expansion to finite injury.

An outline of the paper is as follows.

In §1 we introduce the basic notions of α-recursion theory and α-complexity theory that underlie this paper.

In §2 we present a statement of the α-Union Theorem together with a discussion of the differences between our proof and the McCreight-Meyer one for the case $\alpha = \omega$. While the latter employs a no-injury cancellation construction the former requires a no-injury atop a finite injury cancellation construction. Furthermore, the introduction of finite injury to the proof necessitates a blocking strategy in order to ensure proper convergence.

In §3 we present the actual priority construction that yields an α-recursive k required by the α-Union Theorem. This is followed, in §4, by the verification that the construction is correct.

In §5 two generalizations of the class of (ordinary) primitive recursive functions (ω-Prim) are studied. Though both coincide on ω, they are seen to diverge on all admissible $\alpha > \omega$. The basis of the investigation is a sequence of three propositions (A, B and C) characterizing an arbitrary class, Prim, of functions. Any class which is a model of all three is seen, via an application of the α-Union Theorem, to decompose into an α-hierarchy based on computational complexity. Further, there also exists a single α-recursive bound on the complexity of any function in the class.

The class ω-Prim serves as a model for the three propositions implying the above two consequences. Of the two generalizations, it is seen that one satisfies A but neither B nor C; the second satisfies A and B while that of C is left open. A consequence of the failure of a class to satisfy Proposition C is the nonexistence of an α-analogue ($\alpha > \omega$) to Grzegorczyk's hierarchy for that class. In particular, it is shown that this deficiency holds for the well-known class Prim_0 of Jensen and Karp.

Throughout the paper several open problems are proposed.

1. **Preliminaries.** Let L_α be the collection of sets obtained from Gödel's [4] transfinite hierarchy of constructible sets before α. α is Σ_1 *admissible* if L_α satisfies the replacement axiom schema of ZF for Σ_1 formulae. From now on α is taken as a fixed Σ_1 admissible ordinal.

We employ usual set theoretic notation: $\bigcup A$ for the union of A; $\bigcup_{\varepsilon<\tau} G_\varepsilon$ for the indexed union of G_ε, $\varepsilon < \tau$; $A - B$ for the set difference between A and B; $f|B$ for the mapping f restricted to B; $f[B]$ for the range of $f|B$; $\delta \in B$ for δ an element of B; $A \subseteq B$ for A a subset of B, $A \subset B$ for A a proper subset of B; dom(f), rng(f) for the domain and range of f; and $f: A \to B$ for f a map from A to B.

A partial map $f: \alpha \to \alpha$ is α-partial recursive if its graph has a Σ_1 definition over L_α (with parameters in L_α) and is α-recursive if it is also total on α. A nonempty subset A of α is α-recursively enumerable (α-r.e.) if it is the range of an α-recursive function or, equivalently, the domain of an α-partial recursive function. A is α-recursive if it and its complement (with respect to α) are α-recursively enumerable. Since there exists a one-one α-recursive map from α to L_α, we need only concern ourselves with functions on α and subsets of α.

The main point about any Σ_1 admissible α is that one can perform Δ_1 (α-recursive) recursions in L_α. In particular, one can Gödel number the α-recursively enumerable subsets of α (employing α-recursive pairing $\langle \cdots \rangle$ and projection π_i functions) and, consequently, all the α-partial recursive functions.

We call a subset $A \subset \alpha$ α-bounded (or simply bounded) if there exists a $\beta < \alpha$ so that $\sigma \in A \rightarrow \sigma < \beta$. A is α-finite if it is both α-recursive and α-bounded, alternatively, if A is a member of L_α. A consequence of the definitions of Σ_1 admissibility and α-partial recursiveness is

1.1 Fact. If f: $\alpha \rightarrow \alpha$ is α-partial recursive and A an α-finite subset of dom(f), then $f[A]$ is α-finite.

A key notion in α-recursion theory is that of projecta. The Σ_1-projectum (or simply projectum) of α, α^*, is the least ordinal $\beta \leqslant \alpha$ such that there exists a one-one α-recursive t (referred to as the projection) mapping from α into β. An α-recursive projection map t: $\alpha \rightarrow \alpha^*$ often serves as a vehicle in α-recursive constructions which hinge upon the notion of "priority".

For many Σ_1 admissibles there exist subsets of α that are α-recursively enumerable and bounded (below α), but not α-finite. However, if the bound is small enough, α-finiteness must occur.

1.2 Fact. If $\eta < \alpha^*$ and A is a Σ_1 subset of η, then A is α-finite.

As is often the case α-priority constructions (this paper, in particular) require shorter listings than α^*. One such ordering is supplied by Σ_2 cofinality. For $\gamma \leqslant \alpha$, h: $\gamma \rightarrow \alpha$ is a Σ_2 cofinality function if h is Σ_2 and its range is unbounded in α. The Σ_2 cofinality of α, called $\sigma 2\text{cf}(\alpha)$, is the least $\gamma \leqslant \alpha$ for which there is a Σ_2 cofinality function h: $\gamma \rightarrow \alpha$.

In [1] Blum axiomatizes several properties common to most interesting measures of the complexity of computation on partial recursive functions. A generalization to α-recursion theory of his notion of abstract complexity measure forms a basis of this paper.

1.3 Definition. An *α-computational complexity measure* Φ is an enumeration (in α) of the α-partial recursive functions $\{\phi_\varepsilon | \varepsilon < \alpha\}$ to which are associated the α-partial recursive *α-step counting functions* $\{\Phi_\varepsilon | \varepsilon < \alpha\}$ for which the following axioms hold.

(1) for all $\beta, \varepsilon < \alpha$, $\phi_\varepsilon(\beta)$ is defined if and only if $\Phi_\varepsilon(\beta)$ is defined;

(2) the predicate $M(\varepsilon, \beta, \gamma) \Leftrightarrow \Phi_\varepsilon(\beta) = \gamma$ is α-recursive; and

(3) the α-recursive analogues to the S_n^m-, Universal Function, and Recursion Theorems hold for the enumerations $\{\phi_\varepsilon\}$ and $\{\Phi_\varepsilon\}$ (cf. [10]).

Implicit in the above definition is the capability to retrieve, given any index $\varepsilon < \alpha$, both the function ϕ_ε, in the form of an algorithm, and its α-step counter, Φ_ε. Clearly when $\alpha = \omega$, the definition reduces to that of Blum's. Several illustrations of α-computational complexity measures appear in [7].

Another generalization of a complexity oriented notion is that of α-complexity class.

DEFINITION. For an α-complexity measure Φ and α-recursive function s the *α-complexity class*, C_s^Φ, is the set $\{\phi_\sigma | \phi_\sigma$ is total and $\Phi_\sigma(\beta) \leqslant s(\beta)$ for all but an α-finite set of $\beta\}$.

Hence, C_s^Φ (C_s when Φ is understood) is the set of all α-recursive functions whose α-complexity is bounded by s on all but an α-finite subset of α. Since s is α-recursive, the substitution of bounded for α-finite yields an equivalent definition for C_s^Φ.

Finally, we call a sequence of α-partial recursive functions $\{f_\varepsilon | \varepsilon < \alpha\}$ α-recursively enumerable if there is an α-recursively enumerable set of algorithms so that each function is named at least once.

2. **Discussion.** One objective of this paper is to generalize or "lift" to α the well-known McCreight-Meyer Union Theorem [19]. Namely, that the α-complexity classes of an increasing α-recursively enumerable sequence of α-recursive functions constitute a single α-complexity class.

2.1 α-UNION THEOREM. *Let Φ be an α-computational complexity measure and $\{f_\varepsilon | \varepsilon < \alpha\}$ an α-recursively enumerable sequence of α-recursive functions. Suppose for ε, τ and $\beta < \alpha$, $f_\varepsilon(\beta) < f_\tau(\beta)$, whenever $\varepsilon < \tau$. Then there exists an α-recursive function $k(\beta)$ such that $C_k^\Phi = \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi$.*

First consider $k(\beta) = f_\beta(\beta)$ as a possible candidate for the bounding function. Observe that for $\varepsilon < \alpha$ the set $\{\beta | k(\beta) < f_\varepsilon(\beta)\}$ is α-recursive and bounded (by ε), hence α-finite. Therefore, $C_{f_\varepsilon} \subseteq C_k$ and $\bigcup_{\varepsilon<\alpha} C_{f_\varepsilon} \subseteq C_k$ for all $\varepsilon < \alpha$.

However, the opposite inclusion may not necessarily hold. For there might exist some α-recursive ϕ_ν for which $\Phi_\nu(\beta) \leqslant k(\beta)$ on all but an α-finite set but for each $\varepsilon < \alpha$, $\Phi_\nu(\beta)$ exceeds $f_\varepsilon(\beta)$ for an unbounded set of β. Therefore ϕ_ν would be in C_k but *not* in the union of all the C_{f_ε}.

One possible remedy is seen in the McCreight-Meyer proof of the (ω-) Union Theorem. There a recursive function k is developed via a construction founded upon a priority mechanism. Namely, at stage s ($< \omega$) of the construction a guess is established ($\langle s,s\rangle$) that $f_s(n) \geqslant \Phi_s(n)$ almost everywhere (a.e.). Next a search is made through previous guesses ($\langle v,t\rangle$) for those which prove to be incorrect on input s; that is, $f_t(s) < \Phi_v(s)$. If none are found the value of k on s is $f_s(s)$. Otherwise, the incorrect guess with highest priority, $\langle v',t'\rangle$ (lowest value of v'), is replaced by a new guess, $f_s(n) \geqslant \Phi_{v'}(n)$

a.e. ($\langle v',s\rangle$), and the value of $k(s)$ becomes $f_{t'}(s)$.

Now if ϕ_v were not in the ω-union, then it cannot be in C_k. Since $\phi_v \not\in \bigcup_{t<\omega} C_{f_t}$, then for all $t < \omega$, $\phi_v \not\in C_{f_t}$, hence $f_t(n) < \Phi_v(n)$ infinitely often. Consequently, there will be infinite sequences $\{s_i | i < \omega\}$ and $\{\langle v,t_i\rangle | i < \omega\}$ such that $\langle v,t_i\rangle$ is the incorrect guess of highest priority at stage s_i, forcing k below Φ_v at least as often.

On the other hand, ϕ_v in $\bigcup_{t<\omega} C_{f_t}$ implies ϕ_v is in C_k. Since ϕ_v is in the union, there exist n_0 and $t_0 < \omega$ such that $\Phi_v(n) < f_{t_0}(n)$ for all $n > n_0$. Ultimately some stage of the construction must be reached so that all assignments to k will be made either through $f_s(s)$, $s > t_0$ *or* $f_t(s)$, $t > t_0$. The increasingness of the $\{f_i\}$ therefore implies k exceeds Φ_v almost everywhere.

A generalization of the Union Theorem to α necessitates a complete overhaul of the ω proof. Although we have enumerations in α of the functions $\{\phi_\beta\}$, $\{\Phi_\beta\}$ and $\{f_\beta\}$, we cannot use α (as we did ω above) as a basis of our priorities. A difficulty is manifested in the fact that segments bounded below α can be mapped one-to-one onto unbounded (hence, α-infinite) sequences of α. Thus, we are *not* able to claim that if $\{\beta | \Phi_\nu(\beta) > f_\tau(\beta)\}$ is unbounded then ultimately a stage will have to be reached at which the guess $\langle \nu,\tau\rangle$ will have highest priority and therefore be cancelled.

The usual solution to this is to make the priority listing shorter than α with one such vehicle being the Σ_1 projectum α^* of α. This, in fact, forms the basis of the lifts to α of the Blum-Rabin Complexity and the Borodin Gap Theorems in [8]. The key point is that below α^*, α-recursive enumerability is tantamount to α-finiteness. Thus we cancel indices $\varepsilon < \alpha$ only when their images under the Σ_1 projection map are the smallest cancellable. Standard arguments then show that for any $\varepsilon < \alpha$ the collection of stages at which images of indices having higher priority than ε (images below that of ε in α^*) are cancelled is α-finite.

However, this approach will not totally suffice in our situation. For although we know $\Phi_\varepsilon(\beta) > f_\tau(\beta)$ unboundedly often implies that guess $\langle \nu',\tau\rangle$, $\nu' < \alpha^*$, will ultimately be cancelled, another problem arises. Namely, that for any $\tau_0 < \alpha$, there must be a stage σ_0 such that subsequent assignments to k are made *through cancelled guesses* $\langle \nu,\tau\rangle$ *where* $\tau > \tau_0$. In other words, there is no reason that for some τ_0, an α-infinite number of guesses $\langle \nu,\tau\rangle$ may exist for which τ is smaller than τ_0.

We wrestle with this latter problem by formulating k through a *finite-injury-atop-a-no-injury-cancellation-construction*. The latter feature ensures that unboundedly often wrong guesses will ultimately be cancelled (or "popped"). The former ensures that "τ" components of cancelled incorrect guesses never α-infinitely often regress below any $\tau_0 < \alpha$.

However (as demonstrated in [17], [23] and [24]) when injury arguments are lifted to α, one often requires even shorter listings than the projectum. For although one may have α-finitely many injury sets (here, sets of stages at which guesses with fixed priority values $< \alpha^*$ are injured) and that each set may be proven α-finite, the union of these sets may grow without bound. (See Shore [25] for the usual illustration of $\aleph_\omega$.)

Throughout this paper $\mu = \sigma 2\mathrm{cf}(\alpha)$, h the corresponding Σ_2 cofinality function (h: $\mu \to \alpha$) and t the one-one α-recursive projection map (t: $\alpha \to \alpha^*$). The approach we follow is to formulate a construction that implements a strategy first introduced by Sacks and Simpson [21] and later developed and strongly expedited by Shore [22], [23], [24]. The idea is to segment α^* into a chain of blocks having type equal to the Σ_2 cofinality of α. The block associated with $\rho < \mu$, called B_ρ (the ρth block), is simply the initial segment of α^* bounded above by $t \circ h(\rho)$.

2.2 Lemma. *$t \circ h$ projects all of μ into an unbounded sequence in α^*.*

Proof. Let $\delta < \alpha^*$ to find some $\rho < \mu$ where $t \circ h(\rho) > \delta$. Since $\delta < \alpha^*$, by Σ_1 admissibility and the Σ_1-ness of t^{-1}, there is a β_0 in α where $t(\beta') > \delta$ for $\beta' > \beta_0$. Since h is a Σ_2 cofinality map, there is a $\rho < \mu$ such that $h(\rho) > \beta_0$. $\square$

Since our construction is to be α-effective, we require an α-recursive approximation to h. Namely, let

$$h(\beta) = \delta \Leftrightarrow (\exists \sigma_1)(\sigma_2) R(\sigma_1,\sigma_2,\beta,\delta)$$

for α-recursive R. Then the *σth approximation to h* ($\sigma < \alpha$) is defined as

$$h^\sigma(\beta) = \delta \Leftrightarrow (\exists \sigma_1)_{<\sigma}(\sigma_2)_{<\sigma} R(\sigma_1,\sigma_2,\beta,\delta) \\ \& (\forall \gamma)_{<\delta}(\forall \sigma_1)_{<\sigma}(\exists \sigma_2)_{<\sigma} \neg R(\sigma_1,\sigma_2,\beta,\gamma);$$

h^σ is α-recursive (by admissibility) and h is the limit (as $\sigma \to \alpha$) of h^σ.

2.3 Lemma. *For all $\rho < \mu$ there exists a $\sigma_\rho < \alpha$ such that $(\forall \sigma)[\sigma > \sigma_\rho \to h(\rho) = h^\sigma(\rho)]$.* $\square$

For $\sigma < \alpha$ and $\rho < \mu$, $B_\rho(\sigma)$ denotes the *σth approximation to block B_ρ*. Namely, the initial segment of α^* bounded above by $t \circ h^\sigma(\rho)$. An immediate consequence is the eventual stability of each $B_\rho(\sigma)$.

2.4 Corollary. *For all $\rho < \mu$ there is a $\sigma_\rho < \alpha$ such that $(\forall \sigma)[\sigma > \sigma_\rho \to B_\rho = B_\rho(\sigma)]$.* $\square$

3. **Construction.** An α-recursive function k will be defined in terms of a construction given below. Throughout the execution of the construction several sets will be accumulating.

The set K^σ, at stage σ, represents the function k being built. A pair $\langle \beta,\theta \rangle$

is placed into K^{β} at some stage β if and only if $k(\beta) = \theta$. The set I^{σ}, at stage σ, is a collection of encodings of triples $\langle \nu,\kappa,\gamma \rangle$, $\nu,\kappa,\gamma < \alpha$, such that we have made a guess $f_{\kappa}(\beta) \geqslant \Phi_{\nu}(\beta)$ on all but an α-finite set of β.

A triple $\langle \nu,\kappa,\gamma \rangle$ is said to have *priority value* $t(\gamma) < \alpha^*$. If for some $\rho < \mu$ $(= \sigma 2\mathrm{cf}(\alpha))$, $t(\gamma) \in B_{\rho}$, we call the triple a *ρ-triple*, and if $t(\gamma) \in B_{\rho}(\sigma)$ we refer to it as a *ρ-triple at stage σ*. When the first component of triple $\langle \nu,\kappa,\gamma \rangle$ is to be emphasized, we say it is a *Φ_{ν}-triple*.

The set TO^{σ}, at stage σ, consists of triples of guesses which have been rejected. The reason for this is to have sets which increase as $\sigma \to \alpha$ instead of pulsating. The set $TO^{<\sigma}$ is TO^{σ} just prior to stage σ, that is, $TO^{<\sigma} = \bigcup_{\tau<\sigma} TO^{\tau}$. When $\sigma = \alpha$ we denote $TO^{<\sigma}$ by TO. Similarly, for $K^{<\sigma}$, $I^{<\sigma}$, K and I. Finally, if a triple is in $I^{<\sigma} - TO^{<\sigma}$, we say that it is *active at stage σ* or simply *active* when the context is clear.

The construction which computes $k(\eta)$ for $\eta < \alpha$ is defined by transfinite recursion on stages $\sigma < \alpha$. The key ideas are expressed in the following.

At every stage $\sigma < \alpha$, attempts are made to eliminate bad guesses. If no bad ones are discovered then $k(\sigma)$ takes on the value $f_{\sigma}(\sigma)$. Otherwise, for each $\rho < \mu$ $(= \sigma 2\mathrm{cf}(\alpha))$ the incorrect guess $\langle \nu_{\rho},\kappa_{\rho},\delta_{\rho} \rangle$ of highest priority in $B_{\rho}(\sigma)$ is snuffed out. If $f_{\kappa_{\rho}}(\sigma)$ exceeds the complexity of all correct guesses in blocks $B_{\rho'}$, $\rho' < \rho$, then the triple is cancelled or "popped". Otherwise, the triple is "injured" in the hope that at another time, the $\Phi_{\nu_{\rho}}$-triple will be popped. If some triple $\langle \nu_{\rho},\kappa_{\rho},\gamma_{\rho} \rangle$, $\rho < \mu$, does get popped, then $k(\sigma)$ becomes no larger than $f_{\kappa_{\rho}}(\sigma)$; otherwise, $k(\sigma)$ is $f_{\sigma}(\sigma)$.

A more formal exposition is as follows:

Stage 0. Set $I^{0} = TO^{0} = K^{0} = \varnothing$.

Stage σ. Compute the set

$$V = \{\xi | \xi \in I^{<\sigma} - TO^{<\sigma} \,\&\, f_{\pi_2(\xi)}(\sigma) < \Phi_{\pi_1(\xi)}(\sigma)\}.$$

These are the guesses active at stage σ which we currently discover to be incorrect.

If $V = \varnothing$ then set $\theta = f_{\sigma}(\sigma)$; $K^{\sigma} = K^{<\sigma} \cup \{\langle \sigma, \theta \rangle\}$; $I^{\sigma} = I^{<\sigma} \cup \{\langle \sigma,\sigma,\sigma \rangle\}$; $TO^{\sigma} = TO^{<\sigma}$ and go to stage $\sigma + 1$. So far we have guessed correctly. We therefore set $k(\sigma) = f_{\sigma}(\sigma)$ and establish a new guess that $f_{\sigma}(\beta) \geqslant \Phi_{\sigma}(\beta)$ on all but an α-finite set giving it a priority value $t(\sigma)$.

Otherwise, for each $\rho < \mu$ compute

$$\gamma'_{\rho} = \min\{t \circ \pi_3(\xi) | \xi \in V \,\&\, t \circ \pi_3(\xi) \in B_{\rho}(\sigma)\};$$

$$\nu_{\rho} = \min\{\pi_1(\xi) | \xi \in V \,\&\, t \circ \pi_3(\xi) = \gamma'_{\rho}\} \quad \text{and}$$

$$\kappa_{\rho} = \min\{\kappa | \langle \nu_{\rho},\kappa,t^{-1}(\gamma'_{\rho}) \rangle \in V\}.$$

The triple ξ_{ρ} just chosen from V is the one in $B_{\rho}(\sigma)$ with highest priority (i.e. lowest priority value) according to its third component. If more than one member of V is such, we choose the triple appearing earlier in this list.

Set

$$J_\rho = \{\langle \beta_1,\beta_2,\beta_3\rangle | \langle \beta_1,\beta_2,\beta_3\rangle \in I^{<\sigma} - TO^{<\sigma}$$
$$\& t(\beta_3) \in B_{\rho'}(\sigma) \text{ for } \rho' < \rho \,\&\, f_{\beta_2}(\sigma) \geq \Phi_{\beta_1}(\sigma)\}$$

and

$$m_\rho = \sup\{\Phi_{\pi_1(\xi)}(\sigma) | \xi \in J_\rho\}.$$

The set J_ρ is comprised of all active triples which (1) have priority values in earlier blocks than $B_\rho(\sigma)$, and (2) for input σ, $f_{\beta_2}(\sigma) \geq \Phi_{\beta_1}(\sigma)$ (i.e. represents a correct guess). m_ρ exceeds the amount of work performed by these correct guesses.

If $f_{\kappa_\rho}(\sigma) \geq m_\rho$ then set $I^\sigma = I^{<\sigma} \cup \{\langle \nu_\rho,\sigma,\sigma\rangle\}$ and $TO^\sigma = TO^{<\sigma} \cup \{\langle \nu_\rho,\kappa_\rho,\gamma_\rho\rangle\}$. Here we form a new guess that $f_\sigma(\beta) \geq \Phi_{\nu_\rho}(\beta)$ and append a priority value of $t(\sigma)$. In such a situation a Φ_{ν_ρ}-triple is said to be *popped at stage σ*.

If $f_{\kappa_\rho}(\sigma) < m_\rho$ we set $\tau_\rho = \sup\{\pi_2(\xi) | \xi \in J_\rho\} \cup \{\kappa_\rho\}$, $TO^\sigma = TO^{<\sigma} \cup \{\langle \nu_\rho,\kappa_\rho,\gamma_\rho\rangle\}$ and $I^\sigma = I^{<\sigma} \cup \{\langle \nu_\rho,\tau_\rho,\gamma_\rho\rangle\}$. Here τ_ρ is larger than any middle component of a triple in J_ρ. We eliminate our guess that $f_{\kappa_\rho}(\beta) \geq \Phi_{\nu_\rho}(\beta)$ and replace it with the guess that $f_{\tau_\rho}(\beta) \geq \Phi_{\nu_\rho}(\beta)$ on all but an α-finite set. In such a case we say a Φ_{ν_ρ}-triple is *injured at stage σ*. (Observe that the injured Φ_{ν_ρ}-triple retains its priority but only changes its middle component.)

If for some $\rho < \mu$, $f_{\kappa_\rho}(\sigma) \geq m_\rho$, we set θ to the least such $f_{\kappa_\rho}(\sigma)$; otherwise $\theta = f_\sigma(\sigma)$. In either case, set $K^\sigma = K^{<\sigma} \cup \{\langle \sigma,\theta\rangle\}$ and $I^\sigma = I^{<\sigma} \cup \{\langle \sigma,\sigma,\sigma\rangle\}$. In the former the value of $k(\sigma)$ is the smallest $f_{\kappa_\rho}(\sigma)$ representing a popped Φ_{ν_ρ}-triple. Otherwise, $k(\sigma) = f_\sigma(\sigma)$. An important point is that at any stage σ, if a Φ_{ν_ρ}-triple is *popped*, then $k(\sigma) < \Phi_{\nu_\rho}(\sigma)$.

This concludes the construction. □

4. **Verification.** Clearly k is a well-defined α-recursive function; for to compute $k(\beta)$, $\beta < \alpha$, we simply run the α-effective construction (which at any stage assigns exactly one value to k), up until stage β. The remainder of the proof is the demonstration that $C_k = \bigcup_{\epsilon<\alpha} C_{f_\epsilon}$.

The central convergence result is

4.1 LEMMA. *For any $\rho < \sigma 2\mathrm{cf}(\alpha)$ there exists a stage $\sigma_\rho < \alpha$ such that for all stages $\sigma > \sigma_\rho$ and $\rho' \leq \rho$ blocks $B_{\rho'}$ are stable and no ρ'-triples are injured or popped.*

PROOF. By induction on ρ'. Assume that for all $\rho' < \rho$ the lemma holds and consider the map m: $\sigma 2\mathrm{cf}(\alpha) \to \alpha$ defined as $m(\delta) \equiv$ the least stage σ_δ at which the block B_δ stabilizes and no δ-triple is either injured or popped at any stage $\sigma > \sigma_\delta$.

By the induction hypothesis m is total on ρ. Furthermore, m is Σ_2 from

$$m(\delta) = \sigma_1 \Leftrightarrow (\sigma_2)[\sigma_2 > \sigma_1 \rightarrow B_\delta(\sigma_2) = B_\delta(\sigma_1) \text{ and at stage } \sigma_2 \text{ no } \delta\text{-triple is injured or popped}] \text{ and } \sigma_1 \text{ is the least such;}$$

and since $B_\delta = t \circ h(\delta)$ is Σ_2, $B_\delta(\sigma)$ is Σ_1. From $\rho < \sigma 2\text{cf}(\alpha)$ and the definition of Σ_2 cofinality, $m[\rho]$ is bounded below α. Since t is Σ_1, by admissibility, we can find a bound on the set of all $\sigma < \alpha$ whose image under t falls below $t \circ h(\rho) + 1$. By Corollary 2.4, there exists a stage after which $B_\rho(\sigma)$ stabilizes to B_ρ. Pick σ_1 to exceed the three aforementioned bounds.

Let $\sigma > \sigma_1$ and $\rho' < \rho$ to see that, if $\langle \nu',\kappa',\gamma' \rangle$ is an active ρ'-triple at stage σ, then $\Phi_{\nu'}(\sigma) \leqslant f_{\kappa'}(\sigma)$. For otherwise, by the details of the construction, some ρ'-triple will either be popped or injured at stage σ, contradicting the role of σ_1 as a bound on such stages.

For any ρ-triple, $\langle \nu,\kappa,\gamma \rangle$ involved in any activity following stage σ_1, there are only two possible situations: One is that it is popped. In this case, the Φ_ν-triple has as its second and third components, the stage σ of the popping. However, since $t(\sigma) > t \circ h(\rho)$ the Φ_ν-triple is popped out of block B_ρ. In the other case, it is injured and therefore just the middle component is altered. However, this new component is chosen so that it *exceeds all middle components of ρ'-triples* ($\rho' < \rho$) *currently active* (and hence forever). Consequently, the next time this Φ_ν-triple sees action, it will have to be popped, and as in the previous case, it will be popped out of block B_ρ.

These two cases tell us that after stage σ_1 any Φ_ν-triple which is a ρ-triple will at most be popped from B_ρ, injured once, or injured once and then popped from B_ρ. Since *at most* two active ρ-triples can have the same priority value (i.e. third components equal), we argue that ρ-triples can only contribute an α-finite amount.

Specifically, for $i = 1,2$ define $P^i_\rho = \{\delta | \delta \in B_\rho$ and i ρ-triple(s) with priority value δ is (are) popped after $\sigma_1\}$. Let $IN^i_\rho = \{\delta | \delta \in B_\rho$ and i ρ-triple(s) with priority value δ is (are) injured after $\sigma_1\}$. Since these sets are α-r.e. and bounded below α^* ($t \circ h(\rho) < \alpha^*$) by Fact 1.2 they are α-finite. For $i = 1,2$, define $p^i_\rho(\delta)$ $[in^i_\rho(\delta)] =$ the stage σ at which a ρ-triple with priority value δ is popped [injured] for the ith time. By the α-effectiveness of the construction, in^i_ρ and p^i_ρ, $i = 1,2$, are α-partial recursive. For $i = 1,2$, since $P^i_\rho \subseteq \text{dom}(p^i_\rho)$ and $IN^i_\rho \subseteq \text{dom}(in^i_\rho)$ by Fact 1.1 both p^i_ρ $[P^i_\rho]$ and in^i_ρ $[IN^i_\rho]$, $i = 1,2$, are α-finite. Hence they are all bounded. Thus letting σ_ρ be the maximum of their bounds, it follows by the above definitions that no ρ'-triple, $\rho' \leqslant \rho$, is either injured or popped past stage σ_ρ. Thus the induction step is complete. □

A consequence of the above proof is

4.2 Corollary. *For any $\rho < \sigma 2\text{cf}(\alpha)$ and any δ in B_ρ, there exists a stage*

$\sigma_0 < \alpha$ *such that for all stages* $\sigma > \sigma_0$,

(i) *all blocks* $B_{\rho'}$, $\rho' \leqslant \rho$, *have stabilized.*

(ii) *no* ρ'-*triples are injured or popped for* $\rho' < \rho$, *and*

(iii) *no* ρ-*triples having priority value less than* δ *are injured or popped.*

PROOF. Let σ_1 be as in the induction step of the proof of Lemma 4.1. The only modifications needed are in the definitions of P^i_ρ, IN^i_ρ, p^i_ρ and in^i_ρ, where we only need concern ourselves with those ρ-triples with priority values less than δ. The remainder of the argument carries over. □

In the next lemma we argue that if k dominates the complexity of an α-recursive function, then there must exist some f_{κ_0} also dominating it.

4.3 LEMMA. $C_k \subseteq \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}$.

PROOF. Suppose $\phi_\nu \in C_k$ to show the existence of an f_{κ_0} where $\Phi_\nu(\beta) \leqslant f_{\kappa_0}(\beta)$ on all but an α-finite set of β. This will imply $\phi_\nu \in C_{\kappa_0}$ and hence $\phi_\nu \in \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}$. Assume to the contrary that this is not the case. That is, for *each* $\kappa < \alpha$ the sets $D^\kappa_\nu = \{\beta | f_\kappa(\beta) < \Phi_\nu(\beta)\}$ are not α-finite. The α-recursiveness of the f_κ implies that these must be unbounded.

We prove that the set $A_\nu = \{\sigma |$ at stage σ a Φ_ν-triple is popped$\}$ is unbounded. By the details of the construction this would imply $\{\sigma | k(\sigma) < \Phi_\nu(\sigma)\}$ is also unbounded. Since we assumed $\phi_\nu \in C_k$ this leads to a contradiction completing the proof of the lemma.

Given a stage $\sigma_1 < \alpha$, we show the existence of a stage $\sigma_2 > \sigma_1$ such that at stage σ_2 a Φ_ν-triple is popped, verifying the unboundedness of A_ν. Without loss of generality, we can assume that at the conclusion of stage σ_1 a Φ_ν-triple $\langle \nu, \kappa, \gamma \rangle$ exists. Let $\rho < \sigma 2\text{cf}(\alpha)$ be such that $t(\gamma) \in B_\rho$. Let σ_0 be the stage obtained from Corollary 4.2 and assume σ_0 is at least as large as σ_1. Since there can be at most a second triple having the same priority value $t(\gamma)$, we can assume that σ_0 is large enough so that if this triple is popped, it has done so by stage σ_0. If in between σ_1 and σ_0 a Φ_ν-triple is popped, we are done. Otherwise, assume $\langle \nu, \kappa_1, \gamma \rangle \in I^{\sigma_0} - TO^{\sigma_0}$ for some $\kappa_1 < \alpha$.

Observe that at stages $\sigma > \sigma_0$ for any ρ'-triple $\langle \nu', \kappa', \gamma' \rangle$, $\rho' < \rho$ active at stage σ, that $\Phi_{\nu'}(\sigma) \leqslant f_{\kappa'}(\sigma)$. For otherwise, a ρ'-triple would be either popped or injured contradicting the choice of σ_0. Similarly, for any ρ-triple with priority value less than $t(\gamma)$. By our original assumption, the set $D^{\kappa_1}_\nu = \{\beta | f_{\kappa_1}(\beta) < \Phi_\nu(\beta)\}$ is unbounded. Thus there must be a smallest $\eta \in D^{\kappa_1}_\nu$ such that $\eta \geqslant \sigma_0$. By the details of the construction, and the above remarks, at stage η, the Φ_ν-triple $\langle \nu, \kappa_1, \gamma \rangle$ will be the ρ-triple with lowest priority value active at stage σ such that $f_{\kappa_1}(\eta) < \Phi_\nu(\eta)$. As in the proof of Lemma 4.1, there are two possibilities.

First, if the value $f_{\kappa_1}(\eta)$ is greater than or equal to all values $\Phi_{\nu'}(\eta)$ where at stage η the $\Phi_{\nu'}$-triple is an active ρ'-triple $\rho' < \rho$. In this case the Φ_ν-triple would certainly be popped.

Second, if the value of $f_{\kappa_1}(\eta)$ is less than some $\Phi_{\nu'}(\eta)$ where the $\Phi_{\nu'}$-triple is an active ρ'-triple, $\rho' < \rho$. By the details of the construction the triple $\langle \nu,\kappa_1,\gamma\rangle$ is ejected (i.e., put into TO^{η}) and replaced by the Φ_ν-triple $\langle \nu,\kappa_2,\gamma\rangle$ where $\kappa_2 > \sup\{\kappa' | \kappa'$ is the second component of an active ρ'-triple $\rho' < \rho$ at stage $\eta\}$. Since $D_\nu^{\kappa_2} = \{\beta | f_{\kappa_2}(\beta) < \Phi_\nu(\beta)\}$ is also unbounded, there must be a least $\beta > \eta$ such that $\beta \in D_\nu^{\kappa_2}$. Therefore, at stage β, the Φ_ν-triple $\langle \nu,\kappa_2,\gamma\rangle$ would surely be popped.

Since both possibilities ultimately lead to popping, we are done. □

Our next result shows that if the α-complexity of an α-recursive function is dominated by at least one f_{ε_0} (and consequently all f_ε, $\varepsilon \geq \varepsilon_0$), it must also be dominated by k.

4.4 LEMMA. $\bigcup_{\varepsilon<\alpha} C_{f_\varepsilon} \subseteq C_k$.

PROOF. Let κ be an index such that $\Phi_\nu(\beta) \leq f_\kappa(\beta)$ on all but an α-finite set of β. We show that $\{\beta | \Phi_\nu(\beta) > k(\beta)\}$ is an α-finite set. Without loss of generality, we will assume that κ is the least such index.

Claim. There exist a stage σ_1 and ordinals $\kappa_1,\gamma < \alpha$ such that (1) $\langle \nu,\kappa_1,\gamma\rangle \in I^{\sigma_1} - TO^{\sigma_1}$ and (2) for all stages $\sigma > \sigma_1$, $\langle \nu,\kappa_1,\gamma\rangle \in I^\sigma - TO^\sigma$ (i.e., after stage σ_1 the Φ_ν-triple $\langle \nu,\kappa_1,\gamma\rangle$ remains active).

To prove the claim we first see that the set of stages at which a Φ_ν-triple is popped is bounded. By the hypothesis, the set $\{\beta | \Phi_\nu(\beta) > f_\kappa(\beta)\}$ is α-finite, hence bounded by some $\beta' < \alpha$. By this and the increasingness of the $\{f_\varepsilon\}$, for all $\lambda \geq \kappa$, the sets $\{\beta | \Phi_\nu(\beta) > f_\lambda(\beta)\}$ are bounded above by β'. Suppose an unbounded sequence of stages $\sigma_1 < \sigma_2 < \sigma_3 < \ldots$ exists so that at each stage σ_i a Φ_ν-triple is popped. Consequently, there will be a sequence of Φ_ν-triples for which the second components take on values σ_i. However, once $\sigma_i > \kappa$ and $\sigma > \beta'$, $\Phi_\nu(\sigma) \leq f_{\sigma_i}(\sigma)$, the popping ceases.

Suppose after stage σ' the third component of the Φ_ν-triple remains fixed at value γ. Let $t(\gamma) \in B_\rho$, $\rho < \sigma 2\mathrm{cf}(\alpha)$, and let σ_ρ be the stage obtained by Lemma 4.1. Since after stage σ', the Φ_ν-triple is always a ρ-triple it follows that it can no longer be injured following stage σ_ρ. Hence, the claim is proven.

Observe that κ_1 of the claim is $\geq \kappa$. For otherwise, the set $\{\beta | \Phi_\nu(\beta) > f_{\kappa_1}(\beta)\}$ would be unbounded by the minimality of κ. As in the proof of Lemma 4.3, there would then be a stage $\beta > \sigma_1$ at which the Φ_ν-triple $\langle \nu,\kappa_1,\gamma\rangle$ is popped or injured contradicting the claim.

Let $t(\gamma) \in B_\rho$ and let σ_ρ be the stage obtained from Lemma 4.1 which bounds injuries and pops resulting from ρ'-triples $\rho' \leq \rho$. Since $\{\beta | \Phi_\nu(\beta) > f_\kappa(\beta)\}$ is α-finite, let σ_2 be a bound. Next set $\sigma' = \max\{\sigma_\rho,\sigma_1,\sigma_2\}$ to see that for all $\sigma \geq \sigma'$, $k(\sigma) \geq \Phi_\nu(\sigma)$. By the details of the construction a value $k(\sigma)$ is assigned in one of two ways. First, if $k(\sigma)$ is $f_\sigma(\sigma)$. Then since $\sigma \geqslant \sigma' \geqslant \sigma_1 \geqslant \kappa_1 \geqslant \kappa$,

$$k(\sigma) = f_\sigma(\sigma) \geqslant f_{\kappa_1}(\sigma) \geqslant f_\kappa(\sigma) \geqslant \Phi_\nu(\sigma) \quad (\text{since } \sigma \geqslant \sigma_2).$$

Second, if $k(\sigma) = f_{\kappa'}(\sigma)$ for some $\kappa' < \alpha$. At stages $\sigma \geqslant \sigma'$, the Φ_ν-triple $\langle \nu,\kappa_1,\gamma \rangle$, as well as all ρ' triples, $\rho' \leqslant \rho$, have already stabilized. Thus, for all $\sigma > \sigma'$, if $\langle \nu',\kappa',\gamma' \rangle$ is one of these triples, $f_{\kappa'}(\sigma) \geqslant \Phi_{\nu'}(\sigma)$. By the construction, if a value is assigned to $k(\sigma)$ by popping a $\tilde{\rho}$-triple $\langle \tilde{\nu},\tilde{\kappa},\tilde{\gamma} \rangle$, then it must be that $\tilde{\rho} > \rho$. Thus again by the construction, $\Phi_\nu(\sigma) \leqslant f_{\tilde{\kappa}}(\sigma) = k(\sigma)$.

Therefore, in both situations $k(\sigma) \geqslant \Phi_\nu(\sigma)$ for $\sigma > \sigma'$. Hence, the α-recursive set $\{ \beta | \Phi_\nu(\beta) > k(\beta)\}$ is bounded and thus α-finite. □

Lemmas 4.3 and 4.4 combine to yield $C_k = \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}$.

5. **Complexity of generalized primitive recursive functions.** In this section we examine two infinite analogues to the ordinary primitive recursive functions. Although these generalizations coincide on ω, we show them to diverge on all admissible $\alpha > \omega$. This investigation is complexity oriented and centers around a well-known application of the Union Theorem.

Our first generalization is that of Jensen and Karp [9] with present formulation due to Gandy.

DEFINITION. A function f: $ON^n \to ON$ is *ordinally primitive recursive* (Prim_0) if it can be obtained from the initial functions

(1) $U_i^n(\bar{\beta}) = \beta_i$, $\bar{\beta} = (\beta_1, \ldots, \beta_n)$, $1 \leqslant n < \omega$, $1 \leqslant i \leqslant n$;

(2) $N(\beta) = 0$;

(3) $S(\beta) = \beta + 1 = \beta \cup \{\beta\}$;

(4) $C(\beta,\gamma,\mu,\eta) = \beta$ if $\mu \in \eta$, γ otherwise; under the operations of

(5) substitution:

(5a) $$F(\bar{\beta},\bar{\gamma}) = G\big(\bar{\beta},H(\bar{\beta}),\bar{\gamma}\big), \ \bar{\beta} = (\beta_1, \ldots, \beta_n), \ \ \bar{\gamma} = (\gamma_1, \ldots, \gamma_n), \quad m,n < \omega;$$

(5b) $$F(\bar{\beta},\bar{\gamma}) = G\big(H(\bar{\beta}),\bar{\gamma}\big);$$

(6) recursion:

$$F(\beta,\bar{\gamma}) = G\big(\bigcup \{F(\mu,\bar{\gamma}) | \mu \in \beta\},\beta,\bar{\gamma}\big), \quad n < \omega, \quad \bar{\gamma} = (\gamma_1, \ldots, \gamma_n).$$

A relation on ordinals is *ordinally primitive recursive* just in case its characteristic function is Prim_0.

It is clear from the above definition that there are at most $\aleph_0$ Prim_0 functions. Nevertheless, this small class has enough power to provide analogues to Kleene's T predicates for formalisms of α-recursion theory.

One such development may be obtained through the Kripke equation calculus (EC) (cf. [14]). Specifically, ϕ_ε is the α-partial recursive function computed in the EC from the finite system of equations E with Gödel number ε ($=$ GN(E)) $< \alpha$. The ordinal $\varepsilon < \alpha$ is called an *α-index of* ϕ_ε. If the equations E contain no constants then $\varepsilon < \omega$.

Let $S^E_{\sigma,\gamma}$ be the α-finite collection of equations derived from set E (according to rules of EC) by stage $\sigma < \alpha$ substituting no constants greater than $\gamma < \sigma$. Let π_i, $i \leqslant n$, and $\langle \cdot,\cdot \rangle$ be the usual Prim_0 projection and pairing functions. Following the usual development (cf. Jensen and Karp [9], Kripke [14], Tugué [28]) one shows existence of a Prim_0 T:

$T(\varepsilon,\beta,\gamma) \leftrightarrow$ the equation with Gödel number $\pi_3(\gamma)$ giving value of

α-partial recursive f for β lies in $S^E_{\pi_1(\gamma),\pi_2(\gamma)}$ where $\text{GN}(E) = \varepsilon$;

and a Prim_0 function, $U(\langle \beta_1,\beta_2,\text{GN}(\text{“}f(\beta) = \delta\text{”})\rangle) = \delta$, both of which combine with an α-min operator to yield a Normal Form Theorem.

There is an inherent uniformity about the Prim_0 functions. Namely, if f: $ON \to ON$ is ordinally primitive recursive, then there is an $\varepsilon < \omega$ such that $f|\alpha$ is α-recursive with index ε for *all* admissibles α. As a consequence, f will always map any admissible α (since α is Prim_0 closed [9]) into itself.

As powerful as they appear, the class Prim_0, when regarded as maps from α to α, lack one property of the ordinary primitive recursives. Namely, they are void of the constant functions $\lambda\beta.\gamma$, for $\omega \leqslant \gamma < \alpha$. In the ω-case constant $\lambda x.n$ is derived via n compositions to successor from the null function. For $\alpha > \omega$, the finiteness of the definitions of the Prim_0 functions precludes the derivations of such functions.

Definition. A function f: $\alpha \to \alpha$ is *α-primitive recursive* (α-Prim) if it can be obtained from the initial functions U_i^n,N,S,C and the constant functions $\{\lambda\beta.\gamma|\gamma < \alpha\}$ from the operations of composition and recursion. A relation on α^n ($n < \omega$) is *α-primitive recursive* just in case its characteristic function is α-Prim.

We define an α-complexity measure Φ based upon the Kripke formalism. Specifically, ϕ_ε is the α-partial recursive function computed within the EC from equations having Gödel number ε; the corresponding step counter is

$$\Phi_\varepsilon(\beta) = \min_\gamma T(\varepsilon,\beta,\gamma), \qquad \beta,\varepsilon < \alpha.$$

It is easily seen that $\Phi = \langle \phi_\varepsilon,\Phi_\varepsilon \rangle$ constitutes an α-complexity measure.

In the following α is some Σ_1 admissible ordinal, Φ the Kripke complexity measure defined above and Prim some generalization to α of the primitive recursive functions.

Proposition A. *Let ϕ be a unary* Prim *function. Then there is a unary* Prim *g such that $\phi \in C_g^\Phi$.*

PROPOSITION B. *For g a unary* Prim *function, $f \in C_g^\Phi$ implies $f \in$* Prim.

PROPOSITION C. *There exists an α-recursively enumerable strictly increasing sequence of* Prim *functions $\{f_\varepsilon | \varepsilon < \alpha\}$ such that every* Prim *function is majorized by some f_ε.*

Let Prim $\models \mathbb{S}$ (Prim $\not\models \mathbb{S}$) denote that Prim is (is not) a model of statement $\mathbb{S}$. The key fact is

5.1 THEOREM. *Let* Prim *be a class of functions defined on α such that* Prim $\models$ *Proposition* A & *Proposition* B & *Proposition* C. *Then*

(i) Prim $= \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi$, *where $\{f_\varepsilon\}$ is the sequence of Proposition* C, *and*

(ii) Prim $= C_t^\Phi$, *for some α-recursive t.*

PROOF. (i) Let $\phi \in$ Prim to see Prim $\subseteq \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi$. Since Prim $\models$ Proposition A, there exists $g \in$ Prim such that $\phi \in C_g^\Phi$. By Prim $\models$ Proposition C, $g < f_\varepsilon$ for some $\varepsilon < \alpha$; thus, $\phi \in C_{f_\varepsilon}^\Phi$ and $\phi \in \bigcup_{\phi<\alpha} C_{f_\varepsilon}^\Phi$. For the opposite inclusion, since Prim $\models$ Proposition C, $f_\varepsilon \in$ Prim for all $\varepsilon < \alpha$. Thus by Prim $\models$ Proposition B, $C_{f_\varepsilon}^\Phi$ contains only Prim functions; hence, $\bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi \subseteq$ Prim.

(ii) Let $\{f_\varepsilon\}$ be the α-recursively enumerable strictly increasing sequence of Proposition C. Then, by the α-Union Theorem there exists an α-recursive t such that $C_t^\Phi = \bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi \models$ Prim. □

We next examine which of the various instances of Prim are models of the three propositions.

5.2 LEMMA. (i) α-Prim $\models$ *Proposition* A *for all admissible α, and*

(ii) $\text{Prim}_0 \models$ *Proposition* A.

PROOF. The demonstration that α-Prim $\models$ Proposition A is by induction.

(1) For $U(x) = x$ let $\varepsilon = \text{GN}(\text{“}U(x) = x\text{”})$ and $g_U(\beta) = \langle 1, \beta + 1, \text{GN}(\text{“}U(\beta) = \beta\text{”})\rangle$. By results of Jensen and Karp g_U is Prim_0. Further, in EC, $U = \phi_\varepsilon$ and $\Phi_\varepsilon = g_U$; hence, $U \in C_g^\Phi$.

(2) For $N(x) = 0$, let $\varepsilon = \text{GN}(\text{“}N(x) = 0\text{”})$ and $g_N(\beta) = \langle 1, \beta + 1, \text{GN}(\text{“}N(\beta) = 0\text{”})\rangle$.

(3) For $\lambda x.\gamma$, $\gamma < \alpha$, let $\varepsilon = \text{GN}(\text{“}f(x) = \gamma\text{”})$ and $g_f(\beta) = \langle 1, \beta + 1, \text{GN}(\text{“}f(\beta) = \gamma\text{”})\rangle$.

(4) The case $S(x) = x + 1$ is a bit more complicated and we omit details here. Essentially, one defines successor in EC (cf. Kripke [14]) via a set of 13 equations and then shows $g_S(\beta) = \langle m(\beta), \beta + 2, \text{GN}(\text{“}S(\beta) = \beta + 1\text{”})\rangle$, where m is Prim_0, is the accompanying α-Prim bound.

Implicit in (5) and (6) is a property of pairing functions that for all $\beta < \alpha$, $\beta > \pi_1(\beta), \pi_2(\beta)$.

(5) Assume $k = \phi_{\varepsilon_k}$, $h = \phi_{\varepsilon_h}$, $\varepsilon_k = \mathrm{GN}(E_k)$, $\varepsilon_h = \mathrm{GN}(E_h)$, $k \in C^\Phi_{g_k}$ and $h \in C^\Phi_{g_h}$ where k,h,g_k and g_h are α-Prim. Let

$$\varepsilon_{k \cdot h} = \mathrm{GN}\big(E_k \cup E_h \cup \{\text{“}f(x) = k(h(x))\text{”}\}\big)$$

(assuming no name-conflicts). Let

$$g_f(\beta) = \Big\langle g_k(h(\beta)) + g_h(\beta), \max\{ g_k(h(\beta)), g_h(\beta)\}, \\ \mathrm{GN}(\text{“}f(\beta) = k \circ h(\beta)\text{”})\Big\rangle.$$

(6) Suppose f is defined by recursion equations,

$$(*) \qquad \text{“}f(0) = \gamma\text{”} \quad (\gamma < \alpha), \qquad \text{“}f(\beta) = h\big(\cup\{f(\delta)|\delta \in \beta\}\big)\text{”}$$

where $\varepsilon_h = \mathrm{GN}(E_h)$ and $h \in C^\Phi_{g_h}$. Let $\varepsilon_f = \mathrm{GN}((*) \cup E_h)$ and $g_f(\beta)$ be defined by

$$g_f(0) = \langle 1, \gamma, \mathrm{GN}(\text{“}f(0) = \gamma\text{”})\rangle,$$

$$g_f(\beta) = \Big\langle \max\Big\{ g_h\big(\cup\{f(\delta)|\delta \in \beta\}\big), \sum\{g_f(\delta)|\delta \in \beta\}\Big\}, \\ \max\Big\{ \cup\{g_f(\delta)|\delta \in \beta\}, g_h\big(\cup\{f(\delta)|\delta \in \beta\}\big)\Big\}, \\ \mathrm{GN}\big(\text{“}f(\beta) = h\big(\cup\{f(\delta)|\delta \in \beta\}\big)\text{”}\big)\Big\rangle.$$

Since all functions mentioned (with the exception of those in (3)) involve no infinite constants, the above carries over for Prim_0. □

We next see that complexity classes bounded by α-Prim functions contain only α-Prim functions. Although this implies Proposition B holds for the ordinary primitive recursive functions, we find that this is not so for Prim_0.

5.3 LEMMA. (i) α-Prim $\models$ *Proposition* B.

(ii) $\mathrm{Prim}_0 \not\models$ *Proposition* B.

PROOF. (i) Since Kripke's $T(\varepsilon,\beta,\gamma)$ predicate above is Prim_0, it is α-Prim for any α. Let $\phi_\varepsilon \in C^\Phi_g$ for some $g \in \alpha$-Prim. Since $\Phi^\alpha_\varepsilon < g$ on all but an α-finite subset of α, let δ_0 be a bound. Then clearly,

$$\phi_\varepsilon(x) \equiv U\Big(\min_{\gamma < \max\{g(x),\delta_0\}} T(\varepsilon,\beta,\gamma)\Big).$$

It follows (from results of Jensen and Karp) that ϕ_ε is α-Prim.

(ii) We prove the existence of a $g \in \mathrm{Prim}_0$ and $f \in C^\Phi_g$ where $f \notin \mathrm{Prim}_0$. Define

$$f(x) = \begin{cases} x, & x \neq 3, \\ \omega + 17, & x = 3, \end{cases}$$

and let $g(x)$ be the Prim_0 function $\langle 1,x,\mathrm{GN}(\text{“}f(x) = x\text{”})\rangle$. Since Prim_0

functions map ω into ω, f cannot be Prim_0; however, it is the case that $f \in C_g^\Phi$. □

One of the more well-known properties of the ω-Prim functions was first discovered by Grzegorczyk [5]. Namely, the existence of an ω-hierarchy, $\bigcup_{n<\omega} \mathcal{E}^n$, for ω-Prim. The Grzegorczyk hierarchy is based on the existence of a strictly increasing r.e. sequence of ω-Prim functions $\{f_e(x) | e < \omega\}$ which majorize the ω-Prim functions. The class $\mathcal{E}^n$, $n < \omega$, is defined as containing successor, zero, projections, f_n and closed under operations of composition and a limited or bounded recursion. As a consequence,

5.4 Corollary. ω-Prim $\models$ *Proposition* C.

Since ω-Prim satisfies all three propositions, by Theorem 5.1,

5.5 Corollary. $\omega\text{-Prim} = \bigcup_{\varepsilon<\omega} C_{f_\varepsilon}^\Phi = C_t^\Phi$ *for some ω-recursive t.*

Our next result implies that for $\alpha > \omega$ an α-hierarchy constructed along the lines of Grzegorczyk is impossible for $\text{Prim}_0 | \alpha$.

5.6 Lemma. *Let α be any admissible $> \omega$. Then there cannot exist a sequence $\{f_\varepsilon | \varepsilon < \alpha\}$ of strictly increasing* $\text{Prim}_0|\alpha$ *functions.*

Proof. Suppose $\{f_\varepsilon | \varepsilon < \alpha\}$ is such a sequence. Let $n_0 \in \omega$ and define m: $\alpha \to \omega$ by $m(\beta) = f_\beta(n_0)$. Since each $f_\beta \in \text{Prim}_0$, $f_\beta(n_0) \in \omega$. By the increasingness of $\{f_\varepsilon\}$, m is a strictly increasing (hence, one-one) map of α into ω. Since $\omega < \alpha$, $m[\omega] < m_0 < \omega$, implying the cardinality of ω is finite. □

We have an immediate

5.7 Corollary. $\text{Prim}_0 \not\models$ *Proposition* C. □

Since Prim_0 is a model of Proposition A, but not of B nor C, Theorem 5.1 cannot be used to obtain an α-hierarchy, $\bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi = \text{Prim}_0 = C_t^\Phi$ for α-recursive t and Kripke measure. We, therefore, leave as open the question of the existence of such. Namely, for all admissible α and arbitrary α-complexity measure Φ (not necessarily that of Kripke) do there exist (1) complexity hierarchies for Prim_0 based upon Φ and (2) an α-recursive t such that Prim_0 is the Φ complexity class of t? If negative answers arise, we would naturally be interested in necessary and sufficient characterizations of α where (1) and (2) hold.

Another open problem is the question of whether or not α-Prim is a model for Proposition C. Namely, does there exist an α-r.e. strictly increasing sequence of α-Prim functions having type α such that each α-Prim function is majorized? If the answer is yes, Theorem 5.1 tells us we have an α-hierarchy $\bigcup_{\varepsilon<\alpha} C_{f_\varepsilon}^\Phi = \alpha\text{-Prim} = C_t$ for α-recursive t.

An obvious approach to the above would be the construction of α-analogues to Grzegorczyk's bounding functions. However, difficulties arise in demonstrating that each f_ε, $\varepsilon < \alpha$, is α-Prim. In particular, when $\varepsilon = \lambda$ is a limit, any generalized f_λ, in some way, incorporates a clause

$$f_\lambda(0,y) = \bigcup_{\delta<\lambda} f_\delta(y+1, y+1).$$

The inherent difficulty lies in the fact that an infinite union is being taken where the universal $m(\varepsilon,x,y) = f_\varepsilon(x,y)$ is not α-Prim (in ε,x and y).

A similar problem arises if one attempts to build the sequence of Proposition C from an arbitrary α-r.e. sequence $\{p_\tau | \tau < \alpha\}$ for α-Prim functions. For instance, the maximizing sequence $\{m_\varepsilon(x)\} = \{\sup_{\tau<\varepsilon} p_\tau(x)\}$ is clearly α-r.e., strictly increasing, and α-Prim majorizing. However, the problem occurs in showing $m_\lambda(x)$ α-Prim for λ a limit. In the ω-case each m_ε is ω-Prim since it *has* a finite definition obtainable by incorporating ω-Prim definitions of m_t, $t < \varepsilon$. Upon passing to the infinite this argument is no longer valid. Consequently, one is again dependent upon a non-α-Prim universal function for the p_τ.

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